MATHEMATICAL EXPLORATION

Pi and the Great Pyramid

By Blair Yochim

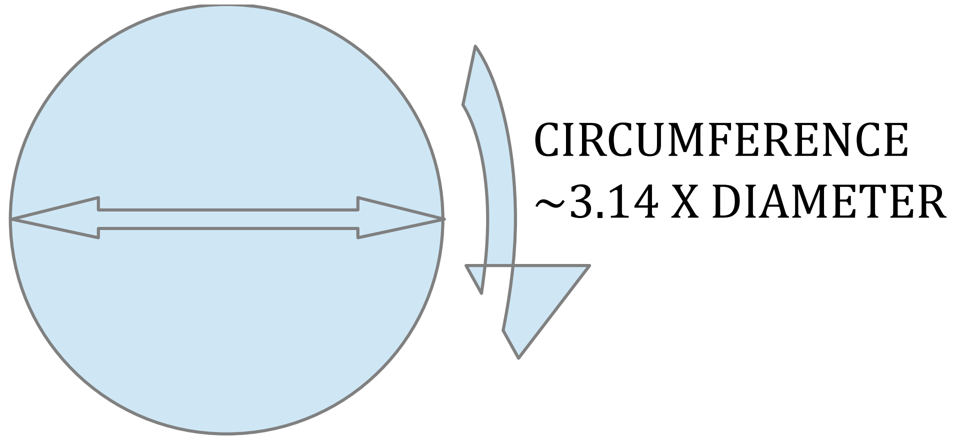
Mr. Yochim passionately specializes in the middle and high school mathematic and science subjects and currently teaches High School special needs students at Mediated Learning Academy in Coquitlam. He received “Teacher of the Year 2006/2007 Awards” from a large local tutoring referral agency and for more than a decade was very active in the BC Government/Science World program called “Scientists and Innovators in the Schools (SIS)” by presenting to over 7,000 students throughout BC. His previous career was a Professional (Electrical) Engineer, is the inventor of a patent on data encryption, and is interested in puzzles, magic, astronomy, space travel, philosophy, vegetarianism, social, environmental, and animal issues.

The Great Pyramid of Giza building in Cairo Egypt (also called Cheops Pyramid or Pyramid of Khufu) was finished about 2560BC, over 4500 years ago. It was the tallest man-made structure for over 3800 years, is one of the wonders of the ancient world, and remains, to this day, a thought-provoking enigma. Throughout history, it has attracted the attention of archaeologists because of the immensity of the ancient building, its construction, and historical significance as well as numerologists who perhaps make false claims.

The value of the mathematical constant pi (represented with the Greek letter π) seems to have been designed into the Great Pyramid to a value of about 3.1419. This value of π was not rediscovered with such accuracy until about 2000 years later. So how did the Egyptians know or use an approximate value of π?! Well, in fact the Egyptians may have incorporated π inadvertently, but not accidentally, into their Great Pyramid with “great” precision without even their awareness of the number! The explanation does not involve numerology or questionable pseudo-scientific Pyramidology but instead involves scientific logic and high school mathematics. Here’s how...

First what is π?

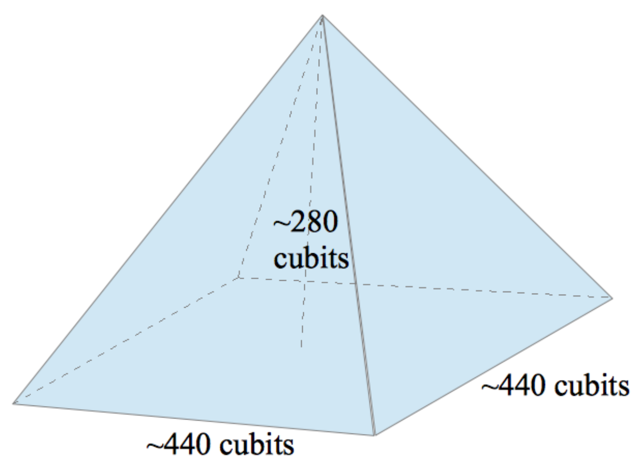
A refresher course: The distance through the middle of a circle is called the diameter. The distance or perimeter around a circle is called the circumference. The circumference around the circle is about 3 times longer than the diameter across the circle, no matter what size of circle is being used. This number is called π and is actually 3.14159265... (considering the length limitations of Vector articles, I’ve chosen not to write the whole thing out) and goes on forever as an irrational transcendental number). This number π is probably one of the simplest to understand mathematical constants, the most used, and probably the most studied.



Measurements of the Great Pyramid :

There have been many publications of differing measurements of the Great Pyramid throughout the last few hundred years by various historical investigators. It is difficult now to get precise measurements of the original Great Pyramid because the outer white limestone layer and the top capstone of the Great Pyramid were removed for other construction projects thousands of years ago and other building blocks have been damaged by factors such as looting, erosion, and dynamite (yes, an English explorer Richard Vyse in the 1800's used dynamite to search for entrances to the Great Pyramid and wasn't the only one to do so!) Also the measurements depend upon the measurement technology used and the professionalism of the archaeology applied. For example, the unit of length used (the cubit) has not been consistent throughout history. So the measurements vary slightly according to the sources. I have, however, tried to consider these variations and have tried to be as accurate as possible in this article.

The Egyptians used a unit of length measurement called the **royal cubit**. A **royal cubit** is the distance from the elbow to the extended middle finger. One royal cubit is about half a metre or more precisely it is 52.37 cm. Historically, there have been some other cubit units used, but they do not apply to the Great Pyramid.



The cap stone white limestone blocks are missing from the top peak of the Great Pyramid, but originally the height was about 146.64 m or 280 cubits high. (One source claims 146.59 m.)

The lengths of the four sides of the base of the Great Pyramid vary from 230.25 m to 230.45 m or about 440 cubits long. (230.25m, 230.36m, 230.39m, 230.45m) Notice that they are all surprisingly close to about 230.35 ± .10 m which is within 10 cm over a distance of 230 m! Remarkable!

If you take half of the perimeter of the Great Pyramid's base

(230.25+230.36+230.39+230.45=921.45m 921.45m ÷ 2 = 460.725m)

and divide it by its height, the result is very close to π!

(460.725m ÷ 146.64m = 3.1419 π=3.14159...)

(Or alternatively if you use the cubit estimate measurements:

440 cubits x 2 ÷ 280 cubits = 22/7 = 3.1429)

There is an Egyptian "Rhind Papyrus" which was written around 1850BC which states π as having a value of (16/9)2 = 3.16049... So how did the Egyptians design their Great Pyramid with a better approximation of π about 700 years earlier? Such accuracy with π wasn't achieved again until Archimedes during the 3rd century BC to obtain π=3.14163.

An explanation of how the Egyptians built their Great Pyramid using π without knowing π:

The key to the mystery of how the Egyptians incorporated π was derived by thinking of another mystery of the Great Pyramid. How were they able to make the lengths of all four of the sides of the Great Pyramid so precise (within 10 cm) to each other over a distance of 230 m?! They could not have used a rope to measure each side of the Great Pyramid to that accuracy because a rope will stretch depending upon the tension and the temperature, especially one 230 m long. They could not have used a long metal rod because the rod will also stretch depending on an inconsistent temperature. Besides, they did not have the means to manufacture a metal rod that long. Another possibility might have been a chain but a chain would have had flexible links. Other modern surveying and measuring technologies did not exist at that time.

Since the four sides of the Great Pyramid are so similar in length, researchers first started to wonder how they did that. I think some Japanese researchers were the first to realize the answer a decade ago or so. The only technology that existed at the time that would give the Egyptians the precision they needed to match all the four side lengths was ... wait for it ... a trundle wheel. If π was involved, it insinuated that a wheel, which has π built-in, was somehow involved. A trundle wheel is just a wheel that is rolled a number of times in order to measure distance. It turns out (pun intended) that a distance is surprisingly very accurately measured with a trundle wheel (small sand or gravel particles are negligible as compared with the size of the curvature of a large trundle wheel when it runs over such obstacles) and it is a technology that is commonly used even today on sports fields and by surveyors. (My classroom has two!)

It is thought that the Egyptians made a trundle wheel carved out of rock with a diameter of exactly 1 cubit (but I don't think archaeologists have found this trundle wheel rock yet to prove this theory). Then they rolled it on flat ground so the trundle wheel rock rolled exactly 140 times around for each of the four sides of the Great Pyramid. Therefore it was rolled a total of 280 times around from one corner of the Great Pyramid to the farthest corner on the other side. This gave them precise locations of the corners of the Great Pyramid and then they started to pile the massive blocks to build the pyramid. They piled the blocks of the Pyramid until they obtained a height of 280 cubits (using the same number 280 as the number of trundle wheel turns from opposite corners of the base). Note that they couldn't easily roll the trundle wheel straight up so instead they counted the same number of cubits to obtain the height.

So they counted 280 turns of the trundle wheel rock from one corner to the opposite corner and measured 280 cubits high from the ground to the peak. Because they used a trundle wheel which has the value of π built into it with the relationship between the circumference and the diameter, the value of π was automatically built into the Great Pyramid without the Egyptians knowing the value of π!

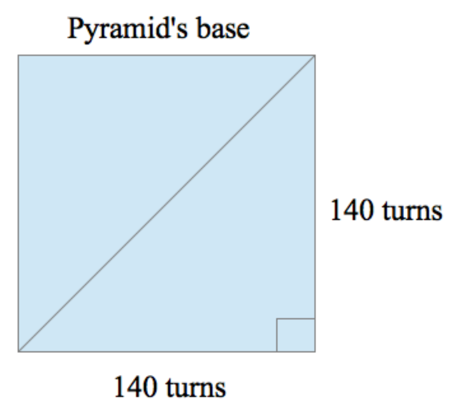
$$\frac{\text{Distance from one corner to the opposite corner}}{\text{Height of the Great Pyramid}} = \frac{280 \times \pi \times 1 \text{ cubit Diameter}}{280 \text{ cubit Diameters}} = \pi$$

But as a mathematician, I must report more details to this story:

Even with a base with four exact measured sides, the Egyptians could have made a base in the form of a rhombus, not a square with 90° corner angles. According to one source, the difference in the distances between the base's diagonal opposite corners is within 17cm which means they indeed obtained corner angles very close to 90°. Another source states that the corner angles are 90° 3' 2", 89° 59' 58", 89° 56' 27", 90° 0' 33" which is an average error of less than 0.03° or 2' (reminder: 1° degree of an angle has 60' minutes and each of those minutes has 60" seconds). So how did the Egyptians obtain near-perfect 90° angles on the corners? The obvious answer would be that they measured the diagonals to be matching lengths or used the Pythagorean Theorem, which if carefully measured on the base perhaps with a trial & error method, produced 90° angles on the corners.

In about 1800BC the Babylonians (located in today's location of Iraq) produced the "Plimpton 322" clay tablet which gave some "Pythagorean Triplet" sets of whole numbers (right-angled triangles with sides that are integers like the 3-4-5 triangle). This was well in advance of Pythagoras in 500BC who now bears his name to the famous formula $a^2 + b^2 = c^2$. But evidence has been claimed to have been found within the Great Pyramid that the Egyptians may have known about some Pythagorean triangles. According to one source they called such triangles "holy triangles". It is questionable though if the Egyptians at the time knew some Pythagorean Triplets because of the lack of accepted written historical evidence.

Suppose that the Great Pyramid designers used a Pythagorean Triplet. One question might be which one would they have used? Could they have used the 140 turns of each side of the Pyramid to obtain a diagonal with an integer value?



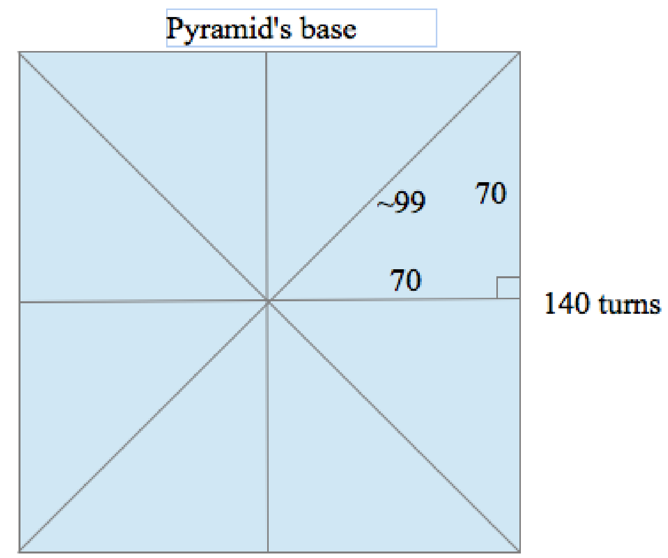
But wait! If you apply the Pythagorean Theorem, you obtain 197.99 turns for the diagonal hypotenuse, not an integer value, but VERY close! Perhaps then, the Egyptians didn't know about Pythagorean Triplets, but instead knew what I call "Pythagorean Isosceles Triplets" which are close to being integers. Upon reflection, of course there never will be exact integer Pythagorean Isosceles Triplets because the hypotenuse will always be √2 times the triangle side and √2 is irrational.

This made me wonder what other Pythagorean Isosceles Triplets exist, so I wrote a spreadsheet program which lists how close the hypotenuse of each set of potential triplets was from an integer. I found more triplets! Here are the closest Pythagorean Isosceles Triplets:

Side (turns)	Side (turns)	Hypotenuse (turns)	Error (turns)
12	12	16.971	0.029
29	29	41.012	0.012
41	41	57.983	0.017
70	70	98.995	0.005
99	99	140.007	0.007
111	111	156.978	0.012
128	128	181.019	0.019
140	140	197.990	0.010
169	169	239.002	0.002

There is a pattern with the triplets. For example, if the 140-140-198 triplet is close, so will the triplet which is half the size, if the original hypotenuse is an even number. A length double the size will also be close. So 70-70-99 is another good possibility. It seems that the Great Pyramid designers had some options on the size of their pyramid. (Also, notice that the Egyptians could've used a larger triplet combination like 169-169-239 which would've made the Great Pyramid even bigger and probably not feasible to build because each dimension would be 20% larger with a 75% increase in the quantity of material needed to build it. Not to say the current size seems at all "feasible" to build!)

Using the 70-70-99 option which has the smallest error in the range used, perhaps they used this design:



Interestingly, notice that 140/99 = 1.4141... which is close to √2 = 1.4142 so were the Egyptians aware of this number? At least one numerologist has made this discovery that the square root of two is also built-in to the Great Pyramid, but this is the case with all of the Pythagorean Isosceles Triplets and of course mathematicians will know that these patterns will naturally always be built-in with a square based pyramid.

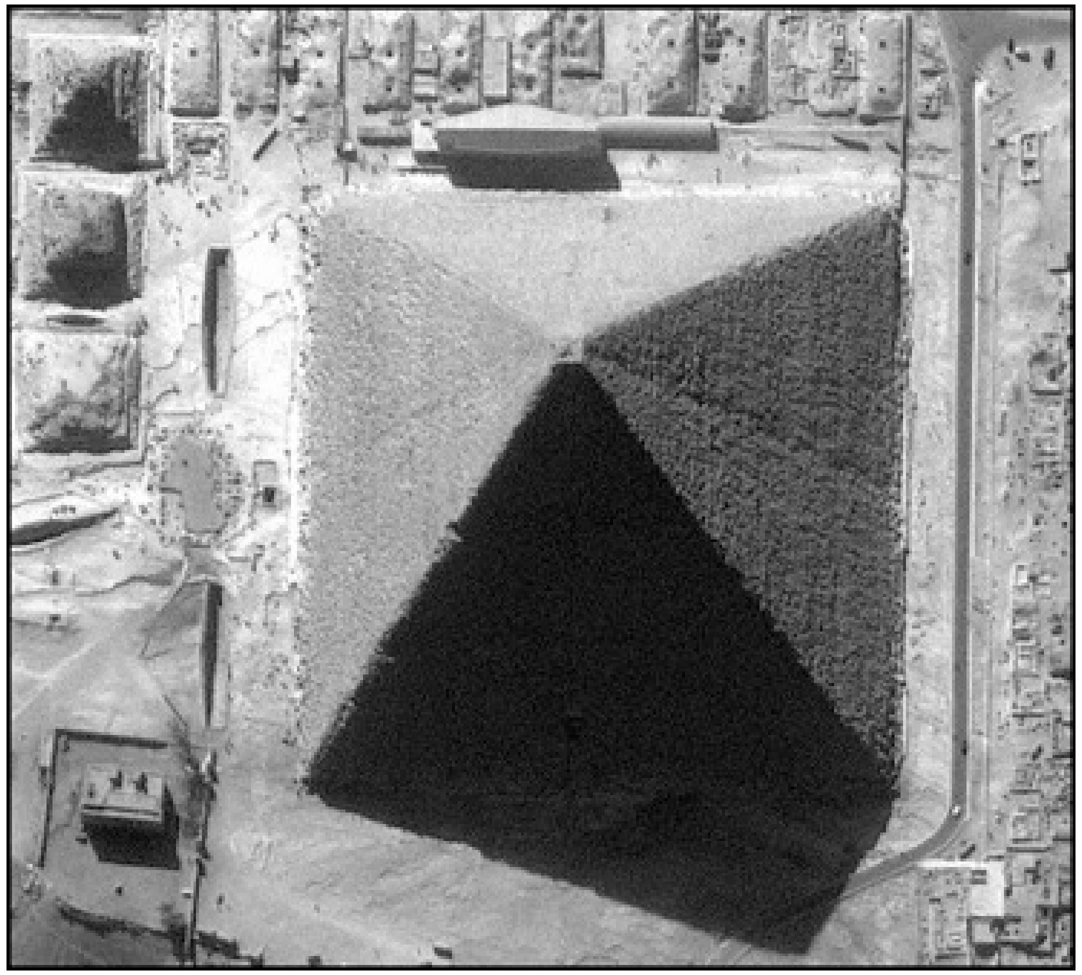
But if the Egyptians used the above design using the 70-70-99 triplet, then they would've used exactly 99 turns of their trundle wheel for the diagonal which would've made the expected 90° right angle inaccurate because of the error in Pythagorean Isosceles Triplets. How close were they with the ~90° angle? Using the Cosine Law: $c^2 = a^2 + b^2 - 2ab\text{Cos } C$

992 = 702 + 702 – 2 x 70 x 70 x Cos Cwe discover that the ~90° is actually C = 90.00585° which

is slightly more than 90°. This would mean that the pyramid would be slightly indented half way on each side towards the middle of the pyramid because the angle in the middle of the pyramid would be double 90.00585° which is 180.0117°.

But this indent actually exists! Under the right particular lighting conditions, shadows show that the Great Pyramid is indeed indented in the middle on each side and apparently is the only pyramid to have this characteristic. So the above 70-70-99 design seems to confirm the hypothesis of the construction of the Great Pyramid!

According to one source, the actual indent on each side averages about 59cm (middle distances through centre reported to be 229.19m & 229.14m). However, one cannot easily directly measure the distance through the centre so this indent "measurement" might be



questionable. How much would this expected indent be with the above 70-70-99 triplet? I calculate the indent on each side would be only about 1.2cm, which is much less than the reported questionable 59cm.

Using a triangle formed by the indent with the error angle of 0.00585° and the hypotenuse length of 70 turns:

sin(0.00585°)=(indent) / (70 turns)   indent = 0.00715 turns

0.00715 turns x π cubits / turn = 0.0225 cubits   0.0225 cubits x 52.37 cm / turn = 1.2 cm.

The above 70-70-99 design seems to have some merit because it explains how π was incorporated within the Great Pyramid with such precision without the Egyptians having to know the value of π or the Pythagorean Theorem and provides a suggested theory as to why the Great Pyramid has indented sides which continues to be a mystery.

So, here was a logical mathematical explanation of how the Egyptians involved π (& √2) with precision in their Great Pyramid without any necessary knowledge of the digits of π or √2. Perhaps someday their 1 cubit diameter trundle wheel tool will be discovered which should support this hypothesis!

This has been an interesting investigation into the mysteries of the Great Pyramid: How did the Egyptians build the Great Pyramid with such precisely equal base sides with limited technologies? How did they involve a surprisingly accurate value of π and √2 in their design thousands of years before such precision was obtained? Why are the sides of the Great Pyramid indented? Using scientific logic and some mathematics, these puzzling questions which have existed for many years have been addressed with plausible theories!

It is historically interesting that the Egyptians possibly may have known of these approximate Pythagorean Isosceles Triplets, perhaps before they discovered exacting integer Pythagorean Triplets such as 3-4-5. But I would be interested in knowing if any of you have heard of the Pythagorean Isosceles Triplets elsewhere and any other applications of them. You may want to consider while you are teaching the Pythagorean Theorem to your own middle or high school classrooms to challenge your students to search for such Pythagorean Isosceles Triplets to find the closest set. This may be a means to show that the hypotenuse of an isosceles triangle will always be an irrational √2 multiple of the sides! *V*